CONVECTIVE INSTABILITY AND VORTICES ON A ROTATING SPHERE

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An interesting problem in gas dynamics is the generation and prolonged existence of ordered vortex structures in a flow with high initial symmetry or even in a fluid at rest. In [1] the problem was considered from the thermodynamic point of view and it was stated that "classical thermodynamics is in essence a theory of the breakdown of structure and it is necessary to add to it the theory of the creation of structure." The appearance of vortex structures in an initially laminar flow, even though the initial steady flow satisfies the conditions of mechanical equilibrium (balance of forces) is due to the development of certain instabilities, which transform the system into a new stable steady state. In the flow created by the vortex structure, the loss of kinetic energy due to dissipative forces must obviously be compensated by a supply of "external" energy.

The classical example of the formation of a vortex structure is the Benárd problem, in which the transition to a new steady state is caused by a convective instability. Application of the theory of hydrodynamic stability, which is based on an analysis of the normal modes, was applied to this problem in [2]. The linear stability theory [2, 3] can be used to find the minimum temperature difference for which there is a steady-state balance between viscous dissipation of kinetic energy and the production of internal energy from buoyancy forces. A similar situation obviously also applies in the formation of steady vortex structures in nature and also in the streamlining of a moving body by a gas.

In the present paper we consider large-scale vortices in the atmosphere of a rotating planet. Development of a convective instability in the centrifugal force field is assumed to be the cause of the generation of steady vortices and latitudinal flow. The dynamics of the overall vortex structure flow on the surface of a uniformly rotating sphere is considered in order to obtain possible steady configurations which are continuously distributed over the entire sphere. In the solutions obtained in the "thin atmosphere" approximation, separate vortices are bounded by separatrices. The rotational energy of the planet serves as the energy source responsible for the prolonged existence of a steady-state structure of rotating vortices and corresponding latitudinal flows.

An important special case is the generation and prolonged existence of the Great Red Spot of Jupiter, which is a vortex in the atmosphere of Jupiter. In contrast to the assumptions and results of previous papers [4-6] on this problem, here the cause of the formation of the Great Red Spot is assumed to be a convective instability, and its well-defined boundary is explained by the fact that it is bounded from the surrounding latitudinal flow field by a separatrix.

Two fundamental assumptions are made in order to obtain analytical solutions. The first is that perturbations of the gravitational field are neglected. The second is that components of the velocity perpendicular to the surface of the planet are neglected.

1. Convective and Centrifugal Instability in Cylindrical Geometry. There are two classical instabilities in the linear theory of waves in a rotating cylinder: in general the convective instability [3], and also the centrifugal instability (the Rayleigh–Taylor instability [7]), when the perturbation is axially symmetric $\partial/\partial\varphi = 0$ and lies in a meridian plane r, z. The physical interpretation of both instabilities is the same and is due to frozen functions (the entropy S for the convective instability, and the angular momentum $I = rv_\varphi$ for the centrifugal instability) floating upward. The local stability criteria are expressed in terms of the derivative of the frozen functions S and I.

Since the initial configuration does not depend on φ and z, the eigenfunctions of the linear problem have the forms

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$$v_r \sim \sin(m\varphi + \omega t)\sin kz,\ \tilde{v}_\varphi,\ \tilde{\rho},\ \tilde{p} \sim \cos(m\varphi + \omega t)\sin kz,$$
$$\tilde{v}_z \sim \sin(m\varphi + \omega t)\cos kz,$$

where the coefficients are functions of r which are to be determined.

We use the system of equations of dissipationless gas dynamics

$$\partial\rho/\partial t + \operatorname{div}\rho\mathbf{v} = 0,\ \rho d\mathbf{v}/dt = -\nabla p - \rho\nabla\Phi,\ dN/dt = 0. \tag{1.1}$$

Here ρ is the density; p is the pressure; Φ is the gravitational potential; $N = p\rho^{-\gamma}$; γ is the adiabatic index. The linearization of this system of equations for an initial steady configuration in which $v_r = 0$, $v_z = 0$, and in which the condition of mechanical equilibrium $v_\varphi{}^2/r = p'/\rho + \Phi'$ is satisfied, reduces to a single equation for f(r) ($f = rv_r/y$, $y = \omega + mv_\varphi/r$) when perturbations in Φ are neglected:

$$\left(\frac{\gamma p y^2}{rs} f'\right)' - \left\{\frac{\rho y^2}{r} - \frac{2\rho v_\varphi}{r^3}\left[(rv_\varphi)' - \frac{2m^2\gamma p v_\varphi}{\rho s r^2}\right] - \frac{p'}{\rho r}\left(\rho' - \frac{m^2\beta p'}{sr^2}\right) - \left[\frac{y^2}{rs}\left(p' - \frac{2m\gamma p v_\varphi}{yr^2}\right)\right]'\right\} f = 0,$$

where $s = \gamma p m^2\beta/\rho r^2 - y^2$, $\beta = 1 + k^2r^2/m^2$, and a prime denotes differentiation with respect to r. In the limit $y \to 0$ this equation reduces to

$$\left(\frac{\rho y^2 r}{\beta m^2} f'\right)' - \left\{\frac{\rho y^2}{r} - \frac{k^2\rho\,(I^2)'}{m^2\beta r^2} - \frac{p'}{\rho r}\left(\rho' - \frac{p'}{c^2}\right)\right\} f = 0 \tag{1.2}$$

($c = \sqrt{\gamma p/\rho}$ is the speed of sound). A necessary local condition for stability then follows:

$$k^2\rho(I^2)'/\beta m^2 r - p'N'/\gamma N > 0. \tag{1.3}$$

In the case of axial symmetry (m = 0) (1.3) gives the stability criterion [8] $\rho(I^2)'/r^3 - p'N'/\gamma N > 0$, for both the convective and centrifugal instability.

For azimuthal perturbations lying in the r, φ plane, we obtain for $k \to 0$ the stability condition $-p'N' > 0$. In this case only the convective instability remains.

In the model of a rotating disk, where the velocity lies in the r, φ plane and the centrifugal force is balanced in the steady state by the pressure gradient, (1.2) can be written in the form

$$(\rho r y^2 f')' - m^2\left(\rho y^2/r - \rho'\nu_\varphi^2 + \rho r\nu_\varphi^4/c^2\right) f = 0 \quad (\nu_\varphi = v_\varphi/r). \tag{1.4}$$

In this case the condition for convective stability $\nu_\varphi{}^2(\rho' - \rho r\nu_\varphi{}^2/c^2) > 0$ implies there is an instability in a uniform rotating field $\rho' = 0$ if one does not take the limit $c^2 \to \infty$.

We put ρ = const, ν_φ = const, c^2 = const in order to estimate the rate of growth (the increment) of the instability. Then the solution of (1.4) satisfying the boundary condition f(R) = 0 will be the Bessel function $J_m(x_{mn}r/R)$, where x_{mn} are the roots of $J_m(x)$. The eigenfrequencies are given by $\omega = -m\nu_\varphi \pm imR\nu_\varphi{}^2/cx_{mn}$. Therefore, the increment increases with m, decreases with n, and is proportional to $v_\varphi{}^2$.

2. Stability of a Rotating Axially Symmetric Layer. Consider a rotating layer of gas in the steady state, bounded by two arbitrary axially symmetric surfaces and in the presence of a gravitational force $\nabla\Phi$. The perturbed motion of the gas is described by the system (1.1), provided that dissipative processes are neglected. We take the axis of rotation to be the z axis of a cylindrical system of coordinates r, φ, z, and introduce the orthogonal curvilinear coordinates s, φ, n, where the distances n and s are measured along the normal to the layer and in the cross section of the meridian plane. Then

$$dn = dz\cos\alpha - dr\sin\alpha,\ ds = dz\sin\alpha + dr\cos\alpha,\ d\mathbf{r}^2 = ds^2 + r^2d\varphi^2 + dn^2$$

(α is the angle between the unit vectors $\mathbf{e}_s$ and $\mathbf{e}_r$, such that $dr/ds = \cos\alpha$). Assuming that in the initial steady state the velocity has a single component $v_\varphi(\alpha)$, and the equation of mechanical equilibrium is satisfied

$$\partial p/\partial s + \rho\partial\Phi/\partial s = (\rho/r)\,v_\varphi^2\cos\alpha, \tag{2.1}$$

and finally assuming that the perturbed velocity does not have a normal component $v = (v_s, v_\varphi)$ and that the gravitational field is unperturbed, the linearized equations of motion for the perturbations $\tilde{v}_s \sim \sin(m\varphi + \omega t)$, $\tilde{v}_\varphi$, $\tilde{p}$, $\tilde{\rho} \sim \cos(m\varphi + \omega t)$ reduce to a single equation for $f = rv_s$:

$$\left(\frac{\rho r c^2}{s} f'\right)' - \left\{\frac{\rho}{r} - \frac{2v_\varphi \cos\alpha}{s}\left(\frac{p'}{y} - \rho I'\right) + \frac{rp'}{\rho s}\left(\rho' - \frac{\rho I'}{yr^2}\right) + \frac{1}{y}\left(\frac{r^2 y p' - \rho c^2 I'}{rs}\right)' + \frac{p'}{y^2}\frac{p' - c^2\rho'}{\rho r s}\right\} f = 0. \quad (2.2)$$

Here $y = \omega/m + \nu_\varphi$, $s = m^2(c^2 - r^2y^2)$, $\nu_\varphi = v_\varphi/r$, $I = rv_\varphi$. In the limit $y \to 0$ we have from (2.2)

$$(\rho r f')' - \left\{\frac{m^2\rho}{r} + \frac{p'}{y^2}\frac{p'/c^2 - \rho'}{\rho r}\right\} f = 0. \quad (2.3)$$

And thus when the local condition

$$p'(p'/c^2 - \rho') > 0 \quad (2.4)$$

is satisfied there is a convective instability which leads to a vortex structure inside the rotating gas layer. Physically the instability is interpreted as entropy floating upward in the force field $\mathbf{F} = r\nu_\varphi^2 \cos\alpha \mathbf{e}_r - \Phi'(s)\mathbf{e}_s$ for $\mathbf{F}\nabla(p\rho^{-\gamma}) > 0$.

For a thin spherical layer in which the gravitational force is directed toward the center of the sphere ($\alpha = \pi/2 - \theta$, $s = R\theta$, $r = R\sin\theta$, θ is the latitude), Eq. (2.1) can be written in the form

$$p'(\theta) = \rho v_\varphi^2 \operatorname{ctg}\theta = \rho\Omega^2 R^2 \sin\theta\cos\theta. \quad (2.5)$$

This equation can be integrated if one assumed the dependence $p \sim \rho^{\gamma_0}$ (see Appendix). Putting $\Omega = \text{const}$, $p = (k/m)\rho T$, we find the latitude dependence of the temperature in the atmosphere of the planet

$$T(\theta) = T(\pi/4) - \frac{1 - 1/\gamma_0}{4k/m}\Omega^2 R^2 \cos 2\theta, \quad (2.6)$$

which depends on the rotational velocity of the atmosphere at the equator ΩR, on the effective molecular mass m, and on the polytropic index γ_0. The parameters m and γ_0 can be eliminated if the law of decrease of temperature with height is known. Indeed, from the equation of equilibrium at the equator $\Omega^2 R = g + (1 - 1/\gamma_0)^{-1} kT'/m$ we have for $\Omega^2 R \ll g$

$$T' = -(1 - 1/\gamma_0)mg/k, \quad T(\theta) = T(\pi/4) + \Delta T \cos 2\theta, \quad (2.6a)$$
$$\Delta T = \Omega^2 R^2 T'/g.$$

This formula is applicable in the troposphere, where the gas density is still sufficiently large. For Earth ($m = 29m_p$, $g = 9.8$ m/sec^2) and for Venus ($m = 44m_p$, $g = 8.9$ m/sec^2) the first equation of (2.6a) gives the correct experimentally measured temperature gradient $T' = -6.5$ and -8 deg/km for the identical value of the parameter $\gamma_0 = 1.235$, which describes the "standard atmosphere" of the earth. Using this value of γ_0 and substituting in (2.6a) the characteristics of the rotating atmospheres of the different planets, we obtain $-T' = 8, 6.5, 3.5, 1.4$ deg/km and $-\Delta T = 3, 35, 15, 200°$, respectively, for Venus, Earth, Mars, Jupiter. Hence it is evident that the average latitude dependence of the temperature in the troposphere of the earth is described to satisfactory accuracy by the second equation of (2.6a). The anomalously large temperature variation $T(\theta)$ on Jupiter is due to its large velocity of rotation.

According to (2.4), a necessary condition for stability of a rotating spherical atmosphere is $v_\varphi^2[\cot\theta v_\varphi^2/c^2 - \rho'(\theta)/\rho] < 0$ and this is the same as the condition for the absence of convection in the centrifugal force field. Putting $p \sim \rho^{\gamma_0}$, $v = \Omega R\sin\theta$ and replacing the argument θ by $x = \cos\theta$, we can write (2.3) as

$$\frac{d}{dx}\left[\rho(1 - x^2)\frac{df}{dx}\right] - \left\{\frac{m^2\rho}{1 - x^2} + \frac{\rho\Omega^2 R^2 x^2}{y^2 c^2}\left(1 - \frac{\gamma}{\gamma_0}\right)\right\} f = 0.$$

The increment of growth is estimated by applying the variational method and using the Legendre polynomials $P_n^m(x)$ as trial functions. Neglecting the variation of ρ and Ω, we obtain

$$\omega = m\Omega \pm i\sqrt{\frac{1-\gamma/\gamma_0}{n(n+1)}\frac{mR\Omega^2}{c}\bar{x}} \quad (\bar{x} \sim 1).$$

Therefore the increment of growth of the instability $\delta \sim \sqrt{1-\gamma/\gamma_0}$ increases with the number of the azimuthal mode m, decreases with the number of the latitude mode n, and is proportional to the square of the rotational angular velocity Ω of the atmosphere of the planet.

For a thin parabolic layer in the gravitational force field $\nabla\Phi = ge_z$ Eq. (2.1) has the form $p'(s)/\rho = r\nu_\varphi^2\cos\alpha - g\sin\alpha$, while (2.2) for an incompressible fluid ($c^2 \to \infty$) gives

$$(\rho r f')' - \left\{\frac{m^2\rho}{r} + \frac{2\nu_\varphi}{y}(\rho\cos\alpha)' - \frac{\rho'\nu_\varphi^2\cos\alpha}{y^2}\left(1 - \frac{g\,\mathrm{tg}\,\alpha}{r\nu_\varphi^2}\right)\right\} f = 0. \tag{2.5a}$$

Let the equation of the parabolic layer be $z = \kappa_0 r^2$, then $\tan\alpha = 2\kappa_0 r$. For constant angular velocity ($v = \text{const}$) the surfaces of constant pressure ($p = \text{const}$) lie on the paraboloids $z = \kappa r^2 + \text{const}$, where $\kappa = \nu_\varphi^2/2g$, and hence $g\tan\alpha/z\nu_\varphi^2 = \kappa_0/\kappa$. Neglecting quantities $\sim\alpha^2$, we obtain

$$(\rho r f')' - \left\{\frac{m^2\rho}{r} + \rho'\left[\frac{2\nu_\varphi}{y} - \frac{\nu_\varphi^2}{y^2}\left(1 - \frac{\varkappa_0}{\varkappa}\right)\right]\right\} f = 0 \tag{2.6a}$$

$$(\varkappa_0/\varkappa = \text{const},\ y = \text{const}).$$

If the free surface $s = s_R$ is maintained in the initial steady state by the atmospheric pressure of the gas with density $\rho_e \ll \rho$, then (2.6a) and the boundary condition at $s = s_R$

$$r\frac{f'}{f} = \frac{2\nu_\varphi}{y} - \left(1 - \frac{\varkappa_0}{\varkappa}\right)\frac{\nu_\varphi^2}{y^2} \tag{2.7}$$

are satisfied by the function $f \sim s^m$ for an arbitrary dependence $\rho(s)$. This leads to the dispersion relation

$$\omega/\nu_\varphi = 1 - m \pm \sqrt{1 - m(1 - \varkappa_0/\varkappa)}. \tag{2.8}$$

For $\kappa > \kappa_0$, when the angular velocity of rotation ν_φ exceeds $\sqrt{g\tan\alpha/r}$, an instability develops with an increment which increases with the number of the azimuthal mode m.

Analogous instability of azimuthal perturbations in a rotating plasma cylinder retained in equilibrium by a magnetic field has been considered in [8-10]. The equations for the corresponding MHD problem can be obtained from (2.6a)-(2.8) if one sets $\kappa_0 = 0$. The "boundary" instability considered here in a rough model of a parabolic layer with constant thickness can lead to a system of vortices, as observed experimentally [4].

In order to estimate the stabilizing effect of the viscosity, we consider the incompressible rotation of a fluid inside the square $0 < x < \ell$, $0 < z < \ell$ described by the stream function $\Psi = A\cos\pi x/\ell\cdot\cos\pi z/\ell$. It follows from the equation $\partial v/\partial t = \nu\Delta v$ (ν is the kinematic viscosity) that the damping decrement is $\delta_1 = \pi^2\nu/\ell^2$. Putting $\delta = R\Omega^2/c$, $R \sim \ell$, we find that stabilization becomes significant when $\delta_1 > \delta$, that is $\Lambda = \Omega^2\ell^3/\nu c < \pi^2$. The dimensionless number Λ is equal to the product of the Mach number M and Reynolds number Re. In particular, for water it follows from these estimates that an instability can develop when $2\pi\Omega > 70\ell^{-3/2}$. However, it should be kept in mind that factors not taken into account here, such as supercriticality, can lead to an increase in the value of Ω necessary for the development of an instability.

3. Steady Vortices in a Rotating Gas. The existence of steady vortices is of great interest in a wide class of problems in gas dynamics. In recent years several papers have been published in which this problem is considered in connection with the construction of dynamical models of the Great Red Spot of Jupiter [5, 6] based on its soliton structure (Rossby soliton). In the present paper the theory of normal modes is used to treat the solution of the general problem for the possible configurations of vortices and the corresponding latitude flows in the atmosphere of a rotating planet with a smooth axially symmetric surface. The use of symmetry arguments and the "thin atmosphere" approximation considerably simplify the problem and reduce it to the solution of a single equation for Ψ, containing an arbitrary function $\mathscr{P}'(\Psi)$. For a suitable choice of $\mathscr{P}(\Psi)$ this equation becomes linear, which leads to a class of exact analytical solutions.

The system of gas-dynamical equations in the absence of dissipation can be written in the following form in a coordinate system rotating with a constant value of Ω:

$$\frac{\partial \mathbf{v}}{\partial t} + [\operatorname{curl}\mathbf{v} + 2\boldsymbol{\Omega}, \mathbf{v}] = -\frac{\nabla p}{\rho} - \nabla\left(\Phi + \frac{v^2}{2} - \frac{1}{2}[\boldsymbol{\Omega}\mathbf{r}]^2\right),$$
$$\frac{\partial \rho}{\partial t} + \operatorname{div}\rho\mathbf{v} = 0, \quad \frac{dS}{dt} = 0. \tag{3.1}$$

Using the thermodynamic identity $dW = dp/\rho + TdS$, we find for steady motion ($\partial/\partial t = 0$):

$$[\operatorname{curl}\mathbf{v} + 2\boldsymbol{\Omega}, \mathbf{v}] = -\nabla\mathscr{P} + T\nabla S, \ \operatorname{div}\rho v = 0,$$
$$\mathscr{P} = W + \Phi + v^2/2 - [\boldsymbol{\Omega}\mathbf{r}]^2/2. \tag{3.2}$$

Since $\mathbf{v}\nabla S = 0$ according to (3.1), it follows from (3.2) that $\mathbf{v}\nabla\mathscr{P} = 0$, and hence the function $\mathscr{P}$ is constant along a streamline of the fluid.

We introduce the orthogonal coordinate system x^i, $dr^2 = g_{ik}dx^i dx^k$ such that the lines x^1 and x^2 lie on the surface Σ under consideration, while the coordinate x^3 is measured along the normal $\mathbf{n}$ to this surface. Then assuming that $\mathbf{vn} = 0$ and that the depth of the layer of liquid (or gas) on the surface Σ is small, we have $v^3 = 0$, $\rho\sqrt{g}v^1 = -\partial\Psi/\partial x^2$, $\rho\sqrt{g}v^2 = \partial\Psi/\partial x^1$, $\operatorname{rot}\mathbf{v} = \mathbf{e}_3\operatorname{div}\nabla\Psi/\rho$, where g is the determinant of the metric tensor g_{ik}. The condition of balance of tangential forces leads to the equation

$$\frac{1}{\rho}\operatorname{div}\frac{\nabla\Psi}{\rho} + 2\frac{\boldsymbol{\Omega}\mathbf{e}^3}{\rho} = \mathscr{P}'(\Psi) - TS'(\Psi). \tag{3.3}$$

Assuming further that the fluid is incompressible with ρ = const, and redefining Ψ such that ρ does not appear in it, we find

$$\sqrt{g}v^1 = -\partial\Psi/\partial x^2, \quad \sqrt{g}v^2 = \partial\Psi/\partial x^1,$$
$$\Delta\Psi + 2\boldsymbol{\Omega}\mathbf{e}^3 = \mathscr{P}'(\Psi), \quad \mathscr{P}(\Psi) = p/\rho + \Phi + v^2/2 - [\boldsymbol{\Omega}\mathbf{r}]^2/2 \tag{3.4}$$

[$\mathscr{P}(\Psi)$ is an arbitrary function]. These equations become exact if $\partial/\partial x^3 = 0$, which occurs for a cylindrical disk when $\mathbf{e}^3 = \mathbf{e}_z$ and $\partial/\partial z = 0$. The equation for Ψ is linear if the function $\mathscr{P}'(\Psi)$ is linear. The function Ω is the angular velocity of the coordinate system in which the flow is steady ($\partial/\partial t = 0$).

For a rotating disk ($x^1 = r$, $x^2 = \varphi$, $x^3 = z$, $\mathbf{e}^3 = \mathbf{e}_z$, $\boldsymbol{\Omega} = \Omega\mathbf{e}_z$) the equations (3.4) become:

$$rv_z = -\partial\Psi/\partial\varphi, \quad v_\varphi = \partial\Psi/\partial r,$$
$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial r^2} = \mathscr{P}'(\Psi) - 2\Omega, \quad \mathscr{P}(\Psi) = \frac{p}{\rho} + \Phi + \frac{v^2}{2} - \frac{\Omega^2 r^2}{2}.$$

For the linear function $\mathscr{P}'(\Psi) = A + B\Psi$ we have

$$\Delta\Psi - B\Psi = A - 2\Omega. \tag{3.5}$$

We seek a solution of (3.5) in the form $\Psi = F(r) + f(r)\cos m\varphi$, which leads to the pair of equations $\frac{1}{r}\frac{d}{dr}r\frac{dF}{dr} - BF = A - 2\Omega$, $\frac{1}{r}\frac{d}{dr}r\frac{df}{dr} - \left(\frac{m^2}{r^2} + B\right)f = 0$. If B is determined from the boundary condition $f(R) = 0$, then $B = -k^2 = -x_{mn}{}^2/R^2$, where x_{mn} are the roots of the Bessel function $J_m(x)$. We then have a solution which is bounded for $r < R$, and which can be expressed in terms of Bessel functions $\Psi = (A - 2\Omega)/k^2 + \alpha J_0(kr) + \lambda J_m(kr)\cos m$, where α and λ are arbitrary constants characterizing the amplitudes of the "latitude" flows and the vortices. In the case considered here, the solution in the rotating coordinate system differs from the solution in a fixed coordinate system [8] only by the addition of a constant to Ψ.

In the case of a rotating sphere, we use the spherical coordinates $x^1 = \theta$, $x^2 = \varphi$, $x^3 = r$, $\mathbf{e}^3 = \mathbf{e}_r$ then for $r \simeq R$ we have $v_\varphi = \partial\Psi/\partial\theta$, $\sin\theta v_\theta = -\partial\Psi/\partial\varphi$.

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} + 2\Omega R\cos\theta = \mathscr{P}'(\Psi). \tag{3.6}$$

It is necessary to solve (3.6) in order to obtain the distribution of steady vortices on the surface of the sphere. A class of exact analytical solutions of (3.6) can be obtained for the linear function $\mathscr{P}'(\Psi) = A + B\Psi$. In this case,

$$\Psi = F(\theta) + f(\theta)\cos m\varphi. \tag{3.7}$$

Here F and f are the solutions of the systems of equations

$$\frac{d}{dx}(1-x^2)\frac{dF}{dx}-BF=A-2\Omega Rx, \quad \frac{d}{dx}(1-x^2)\frac{df}{dx}-\left(\frac{m^2}{1-x^2}+B\right)f=0, \tag{3.8}$$

and $x = \cos\theta$. The function $F(\theta)$ describes the latitude flow, while $f(\theta)$ is proportional to the angular velocity of the vortex at its center, that is at the elliptic singular point of the family of streamlines $\Psi = \text{const}$. A solution of the system (3.8), which depends essentially on φ and which is continuous and single-valued, exists when $B = -n(n+1)$, $n = 1, 2, 3, \ldots$ When $n = 1$ the solution of the first equation is singular and hence regular solutions exist on the entire surface of the sphere only when $n \geq 2$. Then

$$\Psi=\frac{A}{n(n+1)}+\frac{2\Omega R\cos\theta}{2-n(n+1)}+aP_n(\cos\theta)+\lambda P_n^m(\cos\theta)\cos m\varphi, \tag{3.9}$$

where P_n and P_n^m are the Legendre polynomials and associated Legendre functions; a and λ are arbitrary constants determining the amplitudes of the zonal flows and the vortices, respectively. Unlike the case of a plane disk, here there is a term $\sim\cos\theta$ with a coefficient depending on ΩR and on n.

Let the center of the vortex be at the point $\varphi=\varphi_0$, $\theta=\theta_0$. Then in the neighborhood of φ_0, θ_0 we have $\Psi=k[\sin^2\theta_0(\varphi-\varphi_0)^2+\eta^2(\theta-\theta_0)^2]$ where $\eta = \ell_\varphi/\ell_\theta$ is the ellipticity of the vortex. Since $d\varphi/dt = \partial\Psi/\partial\theta$, we have $R\sin\theta_0 d\varphi/dr = 2k\eta^2(\theta-\theta_0)^2$, $\eta^2(\theta-\theta_0)^2 = \Psi/k - \sin^2\theta_0(\varphi-\varphi_0)^2$. Hence $R\sin\theta_0 d\varphi/dt = 2k\eta\sqrt{\Psi/k-\sin^2\theta_0(\varphi-\varphi_0)^2}$. Putting $x = R\sin\theta_0(\varphi-\varphi_0)$, $y = R(\theta-\theta_0)$, we obtain the equation of an ellipse $x^2/\ell_\varphi^2 + y^2/\ell_\varphi^2 = R^2\Psi/k\ell^2 = 1$. Substituting this equation into the equation for $d\varphi/dt$ and integrating, we find the period

$$T=\frac{2R}{k\eta}\int_0^1\frac{d\xi}{\sqrt{1-\xi^2}}=\frac{\pi R}{k\eta}.$$

The angular velocity of the vortex near its axis is then $\nu_* = 2\pi/T = 2k\eta/R$ or, since $2k\sin^2\theta = \partial^2\Psi/\partial\varphi^2$,

$$\nu_*=-\frac{\eta}{R}\frac{m^2}{\sin^2\theta}\lambda P_n^m(\cos\theta)\cos m\varphi. \tag{3.10}$$

Here the subscript 0 is omitted on θ and φ. In the rotating coordinate system the angular velocity of the center of the vortex is equal to zero, while in the fixed coordinate system it is Ω.

Using (3.9), we can express the parameters λ and a in terms of the given coordinates of the center of the vortex φ_0, θ_0 and its ellipticity η. Using the definition of a singular point $\partial\Psi/\partial\varphi = \partial\Psi/\partial\theta = 0$, we obtain two equations for vortices located at azimuths satisfying $\sin m\varphi = 0$:

$$aP_n'+\lambda(P_n^m)'\cos m\varphi=\frac{2\Omega R\sin\theta}{2-n(n+1)}, \tag{3.11}$$
$$aP_n''+\lambda\left[(P_n^m)''+\frac{m^2\eta^2}{\sin^2\theta}P_n^m\right]\cos m\varphi=\frac{2\Omega R\cos\theta}{2-n(n+1)}$$

(the prime denotes a derivative with respect to θ and the subscript 0 is omitted). The solution of the system (3.11) has the form

$$\lambda\cos m\varphi=\frac{2\Omega R/D}{2-n(n+1)}(P_n'\cos\theta-P_n''\sin\theta),$$
$$a=\frac{2\Omega R/D}{2-n(n+1)}\left\{\left[(P_n^m)''+\frac{m^2\eta^2}{\sin^2\theta}P_n^m\right]\sin\theta-(P_n^m)'\cos\theta\right\},$$

where the determinant $D = [(P_n^m)'' + m^2\eta^2 P_n^m/\sin^2\theta]P_n' - (P_n^m)'/P_n''$. For resonant values of θ and η, where $D \to 0$, the constants λ and a go to infinity.

An interesting case occurs when $(P_n^m)_0' = 0$, when $v_\varphi = \partial\Psi/\partial\theta$ goes to zero at $\theta = \theta_0$ separately for latitude flows and for vortices. In this case,

$$\lambda\cos m\varphi = \frac{2\Omega R}{2-n(n+1)}\frac{\cos\theta-\sin\theta P''_n/P'_n}{(P_n^m)''+m^2\eta^2P_n^m/\sin^2\theta},\quad a=\frac{2\Omega R\sin\theta}{2-n(n+1)}\frac{1}{P'_n}. \tag{3.12}$$

Here $\lambda \to \infty$ at resonance and a remains finite. However, it should be noted that large values of λ (corresponding to large vortex amplitudes) are outside of the region of applicability of our approximation $v_r = 0$.

When there exists a background latitude flow with nodes $v_\varphi(\theta)$, vortices will form at the nodes of the background flow [6], since vortices can exist at these points with arbitrarily small amplitude (in the limit $\eta \to \infty$). But in the absence of such a background flow, vortices can be created at any point of the sphere and as the vortex amplitude λ increases, the amplitude of the background a will also increase, and the background and vortex are described by a single "eigenfunction."

If we assume that vortices are generated together with the corresponding latitude flows, then the rotational velocity of the planet can be obtained from the condition $a \to 0$, $\lambda \to 0$, which according to (3.9) implies that $v_\varphi = -2\Omega R\sin\theta/[2 - n(n + 1)]$. This corresponds to rigid body rotation with an angular frequency $\Omega_0 = 2\Omega/[n(n + 1) - 2]$ in the rotating coordinate system. Hence the relative angular velocity of the center of the vortex and the planet is $-\Omega_0 = 2\Omega/[2 - n(n + 1)] < 0$ $(n \geq 2)$. Therefore, the azimuthally independent solutions $B = -n(n + 1)$ in the limit $a \to 0$, $\lambda \to 0$ include steady configurations rotating rigidly with the angular velocity $-\Omega_0$. The symmetric solution $B = 0$, $n = 0$ describes a nonrotating planet with $\Omega_0 = -\Omega$.

When $(P_n^m)_0' = 0$, the use of the second equation of (3.8) gives for ν_* and the variable part of the stream function Ψ

$$\nu_* = \frac{2m^2\Omega\eta}{2-n(n+1)}\frac{\cos\theta-\sin\theta P''_n/P'_n}{n(n+1)\sin^2\theta-m^2(1+\eta^2)}\quad(\theta=\theta_0); \tag{3.13}$$

$$\Psi = \frac{2\Omega R}{2-n(n+1)}\left\{\cos\theta+\frac{\sin\theta_0}{(P'_n)_0}P_n-\frac{\cos\theta_0-\sin\theta_0(P''_n/P'_n)_0}{n(n+1)-m^2(1+\eta^2)/\sin^2\theta_0}\frac{P_n^m}{(P_n^m)_0}\cos m\varphi\right\}. \tag{3.14}$$

When $m = 1$ we have $P_n^m = -P_n'$, and therefore in the "resonance" case $(P_n'') = 0$ and (3.13) and (3.14) become

$$\nu_* = \frac{2\Omega n}{2-n(n+1)}\frac{\cos\theta}{n(n+1)\sin^2\theta-1-\eta^2}\quad(\theta=\theta_0); \tag{3.15}$$

$$\Psi = \frac{2\Omega R}{2-n(n+1)}\left\{\cos\theta+\frac{\sin\theta_0}{(P'_n)_0}P_n-\right.$$
$$\left.-\frac{\cos\theta_0}{n(n+1)-(1-\eta^2)/\sin^2\theta_0}\frac{P'_n}{(P'_n)_0}\cos\varphi\right\}. \tag{3.16}$$

If it is required that one of the circular streamlines coincide with the equator, then $n = 2, 4, 6, \ldots$, and the flow patterns in the northern and southern hemispheres are different, the singular points are located at the nodes of $P_n''(\theta)$ for $\varphi = 0$ and π, and the ellipticities at these points are related by $\eta_+^2 + \eta_-^2 = 2n(n + 1)\sin^2\theta_0 - 2$. If both of the points are elliptic (vortices) then the separatrix angle dividing the two families of vortices rotating in opposite directions can be estimated from the condition $\Psi'' = 0$ at $\theta = \theta_0$. We then obtain $\cos\varphi_S = (\eta_+^2 - \eta_-^2)/(\eta_+^2 + \eta_-^2)$. The requirement $\partial v_\varphi/\partial\theta = 0$ at $\theta = \pi/2$, which is satisfied for $n = 3, 5, 7, \ldots$, leads to a symmetric flow pattern in the northern and southern hemispheres.

Figure 1 shows the steady flow configuration $\Psi = \text{const}$ with vortices on the surface of the rotating sphere constructed from (3.14) with $m = 3$, $n = 2$, $\theta_0 = 60°$, $\eta = 1/\sqrt{3}$ (northern hemisphere). There are six vortices. The small vortices are bounded by separatrices with hyperbolic points displaced toward the pole, and rotate in the same direction as the planet. The large vortices rotate in the opposite direction.

Figure 2 shows the streamlines $\Psi = \text{const}$ for the configuration (3.16) with $m = 1$, $n = 2$, $\theta_0 = 45°$, $\eta = 1$ (northern hemisphere). Here there is a single vortex which rotates in the direction opposite to the planet with $\nu_* = \Omega/\sqrt{8}$ near the vortex axis. The second elliptic singular point is the displaced center of the zonal flows.

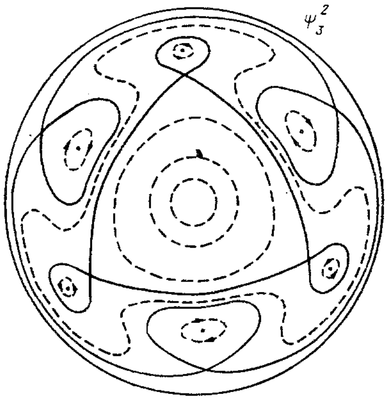

Fig. 1

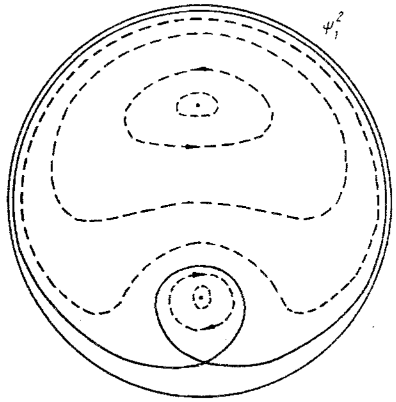

Fig. 2

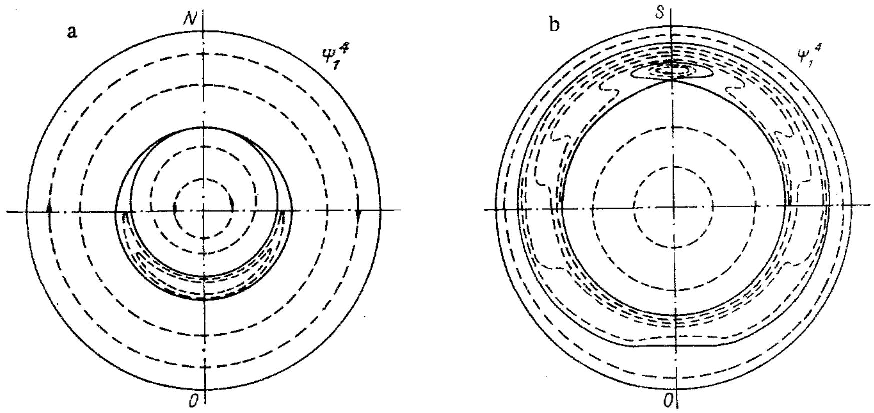

Fig. 3

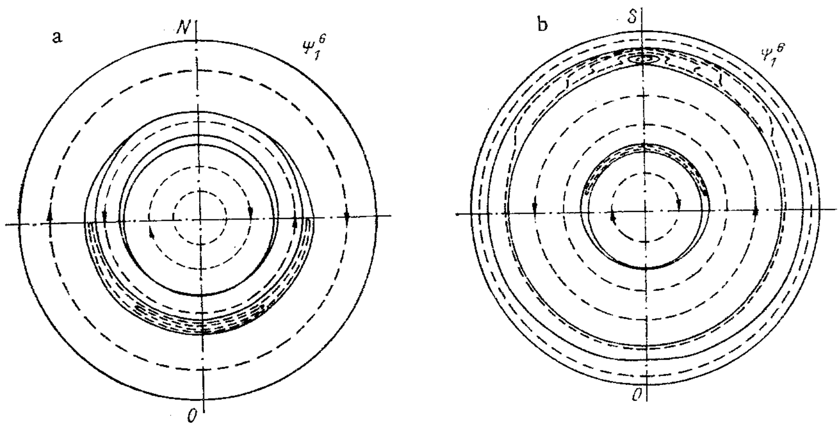

Fig. 4

Figure 3 shows the flow pattern Ψ = const of (3.16) for $m = 1$, $n = 4$, $\theta_0 = 69°$, $\eta = 1$ in the northern (a) and southern (b) hemispheres. The vortex in the southern hemisphere resembles the Great Red Spot of Jupiter in its location, shape, size, and direction of rotation (opposite to the rotation of the planet). Its angular velocity of rotation near the vortex axis is $\nu_* = 0.015\Omega$, and the average angular velocity at the periphery $\nu^* \simeq 0.07\Omega$ is close to the value observed for the Great Red Spot.

Figure 4 shows a similar flow pattern, but for $m = 1$, $n = 6$, $\theta_0 = 76°$. Here the zonal flow has a large number of nodes and there exists a vortex bounded by a separatrix in the southern hemisphere. It is close to the shape of the Great Red Spot and is rotating opposite to the direction of rotation of the planet with $\nu_* = 0.0028\Omega$ near its axis and $\nu^* \simeq 0.03\Omega$ at the periphery. Its size and average rotational velocity are about half as great as for the Great Red Spot on Jupiter.

The steady flow patterns shown in these figures represent a very small class of possible steady flows and result from the linearity of the function $\mathscr{P}'(\Psi)$. The existence of solutions qualitatively describing such exotic phenomena as the Great Red Spot of Jupiter within this small class of solutions indicates that the Great Red Spot can be understood in terms of the theory of normal modes presented here.

Appendix. In general the system of gas-dynamical equations describing the steady state is an incomplete system, and additional equations must be found in order to obtain a unique solution. In many cases the missing equations can be obtained by starting from known stability conditions.

The simplest example is the equilibrium of a plane atmosphere, where there is only a single equation

$$p'(z) = -\rho g \tag{A.1}$$

for two unknown functions: the pressure p and density ρ. Adding the equation of state of an ideal gas $p = (k/m)\rho T$ does not help, since a new function (the temperature T) has been introduced. For this problem it is natural to use the well-known condition for convective stability that the entropy increase with height [3]

$$(p\rho^{-\gamma})' > 0. \tag{A.2}$$

However, the stability condition is an inequality, whereas we need an additional equation in order to obtain a unique solution of the problem. The needed equation can be obtained using the Kelvin hypothesis [11] of equilibrium on the stability boundary. In this hypothesis the inequality (A.2) is replaced by the equation $(p\rho^{-\gamma})' = 0$, which when integrated leads to the constant entropy condition $p\rho^{-\gamma} = \text{const}$. A more accurate description of the structure of the atmosphere results when instead of the adiabatic index γ we use the polytropic index γ_0, which according to (A.2) is smaller than γ. Then as an additional equation we have

$$p\rho^{-\gamma_0} = \text{const.} \tag{A.3}$$

Integrating (A.1) and (A.3) we find the unique solution $T = T_0 - \frac{1-1/\gamma_0}{k/m} gz$, $\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\frac{\gamma_0}{\gamma_0-1}}$, $\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{1}{\gamma_0-1}}$ which contains the single unknown constant γ_0.

The resulting theoretical dependence closely describes the measured variation of temperature and pressure in the dense lower layers of the atmospheres of Earth and Venus for the identical value of the parameter $\gamma_0 = 1.235$.

When a certain height z_1 ($z_1 \approx 11$ and 60 km for Earth and Venus, respectively) is exceeded, the linear decrease in temperature stops and there is a region of constant temperature $T = T_1 = \text{const}$. As follows from the equation of equilibrium (A.1), p and ρ decrease exponentially ($\gamma_0 = 1$): $T = T_1$, $\frac{p}{p_1} = \frac{\rho}{\rho_1} = \exp\left(-\frac{z-z_1}{p_1/g\rho_1}\right)$.

The linear decrease of temperature stops for $z > z_1$ because the atmosphere above z_1 becomes effectively transparent to radiation. That is, its barometric thickness $\ell_1 = p_1/\rho_1 g = kT_1/mg$ is comparable to its optical thickness $\ell = 1/\kappa\rho_1$ which determines the mean path length of photons at height z_1. It then follows that when $z \simeq z_1$ we must have the relation $p = g/\kappa$, where κ is the coefficient of opacity of the atmosphere [12, 13].

The treatment given here is quite general and is also applicable to stellar atmospheres, for which the value of the boundary temperature T_1 is a very important characteristic. For example, it determines the luminosity of the star $L = \pi R^2 \sigma T_1^4$. We can write the equation for the boundary temperature in the form $T_1 = Ag$, $A = m\ell/k$. Assuming that the optical length ℓ is inversely proportional to the effective molecular mass m, we have $A = \text{const}$ and $T_1 \sim g$. The value $A = 22\ \text{K}\cdot\text{sec}^2/\text{m}$ corresponding to the measured values of $T_1 = -57$ and $-75°\text{C}$ for Earth and Venus, gives $T_1 = 6000$ K for the effective surface temperature of the Sun, which is also close to the actual value.

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ASYMPTOTIC OF A SLIGHTLY VISCOUS FLUID FLOW UNDER THE EFFECT OF TANGENTIAL STRESSES ON A FREE BOUNDARY

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Formal asymptotic expansions of the solution of a plane nonlinear stationary problem with a free boundary are constructed for high Reynolds numbers under the assumption that the surface tangential stresses are given and have a finite value. The boundary layer equations near the free boundary are nonlinear, while the principal terms of the asymptotic outside the boundary layer satisfy the Euler ideal-fluid equations. It is shown that the action of the tangential stresses results in the appearance of an additional term equivalent to the surface tension forces in the dynamic boundary condition on the free boundary of a "limit" inviscid flow.

1. A plane nonlinear stationary problem on the motion of an incompressible fluid is considered for the Navier–Stokes equations with vanishing viscosity ($\nu \to 0$) in a domain D bounded by the free surface Γ subjected to tangential and normal stresses given on Γ:

$$(\mathbf{v},\ \nabla)\mathbf{v} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v} + \mathbf{g},\ \operatorname{div}\mathbf{v} = 0; \tag{1.1}$$

$$p - 2\rho\nu\partial v_n/\partial n = p_* + \varkappa\sigma,\quad \rho\nu\mathbf{n}(\boldsymbol{\tau}\cdot\nabla)\mathbf{v} = T,\quad (x,\ z)\in\Gamma; \tag{1.2}$$

$$\mathbf{v}\cdot\mathbf{n} = 0,\ (x,\ z)\in\Gamma. \tag{1.3}$$

Here $\mathbf{v} = (v_x,\ v_z)$, $\mathbf{g} = -g\mathbf{e}_z$, $\mathbf{e}_z = (0,\ 1)$ is the direction of the z axis, ρ is the fluid density, g is the acceleration of gravity, $\sigma = \mathrm{const} > 0$ is the coefficient of surface tension, κ is the curvature of the free boundary Γ ($\kappa > 0$ if Γ is convex outside the fluid); **n** and **τ** are unit vectors of the external normal and the tangent to Γ; p_* is the given pressure on Γ; and T is the tangential stress on Γ [$T = O(1)$ as $\nu \to 0$]. It is assumed that the domain D is not bounded and the behavior of the velocity field at infinity is given.

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